Finite Size XXZ Spin Chain with Anisotropy Parameter $\Delta = \frac{1}{2}$

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Abstract

We find an analytic solution of the Bethe Ansatz equations (BAE) for the special case of a finite XXZ spin chain with free boundary conditions and with a complex surface field which provides for $U_q(sl(2))$ symmetry of the Hamiltonian. More precisely, we find one nontrivial solution, corresponding to the ground state of the system with anisotropy parameter $\Delta = \frac{1}{2}$ corresponding to $q^3 = -1$.

With a view to establishing an exact representation of the ground state of the finite size XXZ spin chain in terms of elementary functions, we concentrate on the crossing-parameter η dependence around $\eta = \pi/3$ for which there is a known solution. The approach taken involves the use of a physical solution Q of Baxter's t-Q equation, corresponding to the ground state, as well as a non-physical solution P of the same equation. The calculation of P and then of the ground state derivative is covered. Possible applications of this derivative to the theory of percolation have yet to be investigated.

As far as the finite XXZ spin chain with periodic boundary conditions is concerned, we find a similar solution for an assymetric case which corresponds to the 6-vertex model with a special magnetic field. For this case we find the analytic value of the "magnetic moment" of the system in the corresponding state.

Dedicated to Rodney Baxter
on the occasion of his 60th birthday.

I. Introduction

It is widely accepted that the Bethe Ansatz equations for an integrable quantum spin chain can be solved analytically only in the thermodynamic limit or for a small number of spin waves or short chains. In this paper, however, we have managed to find a special solution of the BAE for a spin chain of arbitrary length N with N/2 spin waves.

It is well known (see, for example [1] and references therein) that there is a correspondence between the Q-state Potts Models and the Ice-Type Models with anisotropy parameter $\Delta = \frac{\sqrt{Q}}{2}$. The coincidence in the spectrum of an N-site self-dual Q-state quantum Potts chain with free ends with a part of the spectrum of the $U_q(sl(2))$ symmetrical 2N-site XXZ Hamiltonian (1) is to some extent a manifestation of this correspondence.

$$H_{xxz} = \sum_{n=1}^{N-1} \{\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+ + \frac{q+q^{-1}}{4}\,\sigma_n^z\sigma_{n+1}^z\ + \frac{q-q^{-1}}{4}\,(\sigma_n^z - \sigma_{n+1}^z)\}, \tag{1}$$

where $\Delta = (q+q^{-1})/2$. This Hamiltonian was considered by Alcaraz *et al.* [1] and its $U_q(sl(2))$ symmetry was described by Pasqier and Saleur [2]. The family of commuting transfer-matrices that commute with H_{xxz} was constructed by Sklyanin [3] incorporating a method of Cherednik [4].

Baxter's T-Q equation for the case under consideration can be written as [5]

$$t(u)Q(u) = \phi(u+\eta/2)Q(u-\eta) + \phi(u-\eta/2)Q(u+\eta) \tag{2}$$

where $q = \exp i\eta$, $\phi(u) = \sin 2u\, \sin^{2N} u$ and $t(u) = \sin 2u\, T(u)$. The $Q(u)$ are eigenvalues of Baxter's auxilary matrix $\hat{Q}(u)$, where $\hat{Q}(u)$ commutes with the transfer matrix $\hat{T}(u)$. The eigenvalue $Q(u)$ corresponding to an eigenvector with $M = N/2 - S_z$ reversed spins has the form

$$Q(u) = \prod_{m=1}^{M} \sin(u-u_m)\sin(u+u_m). \tag{3}$$

Equation (2) is equivalent to the Bethe Ansatz equations [6]

$$\left[\frac{\sin(u_k+\eta/2)}{\sin(u_k-\eta/2)}\right]^{2N} = \prod_{m\neq k}^{M} \frac{\sin(u_k-u_m+\eta)\sin(u_k+u_m+\eta)}{\sin(u_k-u_m-\eta)\sin(u_k+u_m-\eta)}. \tag{4}$$

Baxter's equation can be interpreted as a discrete version of a second order differential equation[7, 8]. So we can look for its second independent solution $P(u)$ with the same eigenvalue $T(u)$.

In a recent article [9] Belavin and Stroganov argued that the criteria for the above mentioned correspondence is the existence of a second trigonometric solution for Baxter's T-Q equation and it was shown that in the case $\eta = \pi/4$ the spectrum of H_{xxz} contains the spectrum of the Ising model. In this article we limit ourselves to the case $\eta = \pi/3$. This case is in some sense trivial since for this value of η, H_{xxz} corresponds to the 1-state Potts Model. We find only one eigenvalue $T_0(u)$ of the transfer-matrices $\hat{T}(u)$ when Baxter's equation (2) has two independent trigonometric solutions. Solving for $T(u) = T_0(u)$ analytically we find a trigonometric polynomial $Q_0(u)$ the zeros of which satisfy the Bethe Ansatz equations

(3). The number of spin waves is equal to $M = N/2$. The corresponding eigenstate is the groundstate of H_{xxz} with eigenvalue $E_0 = \frac{3}{2}(1-N)$, as discovered by Alcaraz *et al.* [1] numerically.

When the first version of this paper [15] was reported at the Workshop in ANU (February 2000), Baxter showed us his article [16] where he writes about the existence of a simple eigenvalue of the T-matrix for the 8-vertex model for the special case $\mu = \pi/3$. About 30 years ago Baxter discovered [17] that the ground state energy of the XYZ-spin chain which is described by the Hamiltonian

$$H_{xyz} = -\frac{1}{2}\sum_{n=1}^{N}\{J_x\sigma_n^x\sigma_{n+1}^x + J_y\sigma_n^y\sigma_{n+1}^y + J_z\sigma_n^z\sigma_{n+1}^z\}, \tag{5}$$

has an especially simple value $E = -(J_x + J_y + J_z)/4$ for the case $\mu = \pi/3$, when J_x, J_y and J_z satisfy the constraint $J_xJ_y + J_yJ_z + J_zJ_x = 0$. This value was found in the thermodynamic limit only. Later in 1989 [16] Baxter considered a determinant functional relation and found a very simple solution for eigenvalue of T-matrix (see Section II below) for the case $\mu = \pi/3$.

There are however problems with the realization of this simple solution. If we consider, for example, the usual XXZ-spin chain with periodic boundary conditions we will find, solving Baxter's T-Q equation, that the degree of Q(u) is $N/2 + 1$ (N even) and there is no Bethe vector for this simple solution. We believe that for the case without a field, a "physical" solution Q(u) must have degree less than or equal to $N/2$. Baxter informed us that the addition of a special field saves the simple solution and below we find the corresponding Q(u).

At first glance it seems we have a clear picture. For the open chain with special boundary conditions we have central charge $c = 0$ (see for example [18]), so finite-size corrections are absent and we have a simple solution for the ground state. On the other hand, for periodic boundary conditions without a field we have due to Hamer [19] $c \neq 0$, so there is no simple solution.

We wish to stress that essentially only chains with even length were considered. Hamer [19] considered only even N. However, for odd N, one can find [20] a simple eigenvalue $T_0 = (a+b)^N$ for the case when the weights of the 8-vertex model satisfy the condition

$$(a^2 + ab)(b^2 + ab) = (c^2 + ab)(d^2 + ab), \tag{6}$$

which is equivalent the condition $\mu = \pi/3$.

II. Consequences of the existence of the second "independent" periodic solution

This question was considered in article [9]. Here we use a variation more convenient for our goal. This section is mainly due to Baxter [16] (see also [21]).

Let us consider T-Q equation (2) for $\eta = \frac{\pi}{L}$, where $L \geq 3$ is an integer. Let us fix a sequence of spectral parameter values $v_k = v_0 + \eta k$, where k are integers and write $\phi_k = \phi(v_k - \eta/2)$, $Q_k = Q(v_k)$ and $t_k = t(v_k)$. Functions $\phi(u)$, $Q(u)$ and $t(u)$ are periodic

with period π. So the sequences we have introduced are also periodic with period L, i.e., $\phi_{k+L} = \phi_k$, etc..

Setting $u = v_k$ in (2) gives the linear system

$$t_k Q_k = \phi_{k+1} Q_{k-1} + \phi_k Q_{k+1}. \tag{7}$$

The matrix of coefficients for this system has the three-diagonal form

$$M = \begin{vmatrix} -t_0 & \phi_0 & 0 & \dots & 0 & \phi_1 \\ \phi_2 & -t_1 & \phi_1 & \dots & 0 & 0 \\ 0 & \phi_3 & -t_2 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ \phi_{L-1} & 0 & 0 & \dots & \phi_0 & -t_{L-1} \end{vmatrix}. \tag{8}$$

Taking $v_0 \neq \frac{\pi m}{2}$, where m is an integer, we have $\phi_k \neq 0$ for all k.

It is straightforward to calculate the determinant of the $L-2 \times L-2$ minor obtained by deleting the two left most columns and two lower most rows. It is equal to the product $-\phi_1^2 \phi_2 \phi_3 \dots \phi_{L-1}$, which is nonzero, hence the rank of M cannot be less than $L-2$. Here we are interested in the case when the rank of M is precisely $L-2$ and we have two linearly independent solutions for equation (7). Let us consider the three simplest cases L = 3, 4 and 5. Parameter η is equal to $\frac{\pi}{3}$, $\frac{\pi}{4}$ and $\frac{\pi}{5}$ respectively.

For $L = 3$ the rank of M is unity and we immediately get $t_0 = -\phi_2$, $t_1 = -\phi_0$ and $t_2 = -\phi_1$. Returning to the functional form, we can write

$$T_0(u) = t_0(u)/\sin 2u = -\phi(u + \frac{\pi}{2})/\sin 2u = \cos^{2N} u. \tag{9}$$

This is the unique eigenvalue of the transfer-matrix for which the T-Q equation has two "independent" periodic solutions. It is well known (see, for example, [6]) that the eigenvalues of H_{xxz} are related to the eigenvalues $T(u)$ by

$$E = -N\cos\eta + (\frac{T'(\eta/2)}{T(\eta/2)} + \tan\eta)\sin\eta. \tag{10}$$

For the eigenstate corresponding to eigenvalue (9) we obtain

$$E_0 = \frac{3}{2}(1-N). \tag{11}$$

This is the groundstate energy which was discovered by Alcaraz *et al.* [1] numerically.

In the next section we find all solutions of Baxter's T-Q equation corresponding to $T(u) = T_0(u)$. In particular we find $Q(u)$, the zeros of which satisfy the BAE (4).

For $\eta = \frac{\pi}{3}$ and transfer-matrix eigenvalue $T_0(u) = \cos^{2N} u$, T-Q equation (2) reduces to

$$\phi(u + 3\eta/2)Q(u) + \phi(u - \eta/2)Q(u+\eta) + \phi(u+\eta/2)Q(u-\eta) = 0. \tag{12}$$

Keeping $t(u)$ arbitrary, for the moment, we can rewrite (2) as

$$t(u) = \phi(u+\eta/2)\frac{Q(u-\eta)}{Q(u)} + \phi(u-\eta/2)\frac{Q(u+\eta)}{Q(u)}. \tag{13}$$

We also have

$$t(u+\eta)=\phi(u+3\eta/2)\frac{Q(u)}{Q(u+\eta)}+\phi(u+\eta/2)\frac{Q(u+2\eta)}{Q(u+\eta)}. \tag{14}$$

Multiplying these equations together we obtain the fusion relation

$$t(u)t(u+\eta)=\phi(u-\eta/2)\phi(u+3\eta/2)+\phi(u+\eta/2)\tilde{t}(u), \tag{15}$$

where

$$\tilde{t}(u)=\frac{Q(u-\eta)}{Q(u)Q(u+\eta}(\phi(u+3\eta/2)Q(u)+\phi(u-\eta/2)Q(u+\eta)+\phi(u+\eta/2)Q(u-\eta)). \tag{16}$$

In the case under consideration we have $\tilde{t}(u)=0$. Fusion relation (15) reduces to the simple equality

$$t_0(u)t_0(u+\eta)=\phi(u-\eta/2)\phi(u+3\eta/2), \tag{17}$$

which is a kind of inversion relation.

For $L=4$

$$M=\begin{vmatrix} -t_0 & \phi_0 & 0 & \phi_1 \\ \phi_2 & -t_1 & \phi_1 & 0 \\ 0 & \phi_3 & -t_2 & \phi_2 \\ \phi_3 & 0 & \phi_0 & -t_3 \end{vmatrix}. \tag{18}$$

Deleting the second row and the forth column we obtain a minor with determinant $-\phi_0\phi_3(t_0+t_2)$. It is zero when $t_2=-t_0$, i.e., $t(u+\frac{\pi}{2})=-t(u)$. Considering the other minors we obtain the functional equation

$$t(u+\pi/8)t(u-\pi/8)=\phi(u+\pi/4)\phi(u-\pi/4)-\phi(u)\phi(u+\pi/2). \tag{19}$$

This functional equation was used in [9] to find $t(u)$ and show that this part of the spectrum of H_{xxz} coincides with the Ising model. It would be interesting to find a corresponding $Q(u)$.

Lastly for $L=5$, minor M_{35} (the third row and the fifth column are deleted) has determinant $\phi_0\phi_4(t_0t_1+\phi_1t_3-\phi_0\phi_2)$. Setting this to zero we have

$$t(u)t(u+\pi/5)+\phi(u+\pi/10)t(u+3\pi/5)-\phi(u-\pi/10)\phi(u+3\pi/10)=0. \tag{20}$$

It is not difficult to check that in this case all 4×4 minors have zero determinant and that the rank of M is 3. Thus we have two "independent" periodic solutions of Baxter's T-Q equation.

Note that this functional relation (20) coincides with the Baxter-Pearce relation for the hard hexagon model. The connection between (20) and a special value of the rank of the matrix of coefficients for system (7) was remarked upon in [11] by Andrews, Baxter and Forrester.

For general L we obtain the same truncated functional relations that have been obtained in [9] with the same assumptions. Note that for the ABF models [11], which are a generalization of the hard hexagon model, the truncated functional relations have been proved by Behrend, Pearce and O'Brien in [12].

III. Solution of Baxter's Equation for $\eta = \frac{\pi}{3}$ and $T = T_0$

Equation (12) can be rewritten as follows:

$$f(v) + f(v + \frac{2\pi}{3}) + f(v + \frac{4\pi}{3}) = 0, \tag{21}$$

where $f(v) = \sin v \, \cos^{2N}(v/2)\, Q(v/2)$ has period 2π. The trigonometric polynomial $f(v)$ is an odd function so it can be written

$$f(v) = \sum_{k=1}^{K} c_k \sin kv, \tag{22}$$

where K is the degree of $f(v)$. Equation (21) is equivalent to

$$c_{3m} = 0, \qquad m \in Z. \tag{23}$$

The point $v = \pi$ is a zero of $f(v)$ of order $2N+1$, so we obtain

$$(\frac{d}{dv})^i f(v)|_{v=\pi} = 0, \qquad i = 0, 1, \ldots, 2N. \tag{24}$$

For even i this condition is immediate, whereas for $i = 2j-1$ we use (22) to obtain

$$\sum_{k=1, k\neq 3m}^{K} (-1)^k c_k k^{2j-1} = 0, \qquad j = 1, 2, \ldots, N. \tag{25}$$

Our problem is a special case of a more general problem which can be formulated as follows. Given a set of different complex numbers $X = \{x_1, x_2, \ldots, x_I\}$ we seek another complex set $B = \{\beta_1, \beta_2, \ldots, \beta_I\}$ where $\beta_i \neq 0$ for some i, so that

$$\sum_{i=1}^{I} \beta_i P(x_i) = 0 \tag{26}$$

for any polynomial $P(x)$ of degree not more than $N-1$. It is clear that for $I \leq N$ the system B does not exist. If $\beta_1 \neq 0$, for example, the product $(x-x_2)(x-x_3)\ldots(x-x_I)$ provides a counterexample.

Let $I = N+1$. We try the polynomials

$$P_r = \prod_{i=1, i\neq r,}^{N} (x - x_i), \qquad r = 1, 2, \ldots, N. \tag{27}$$

Condition (26) gives $\beta_r P_r(x_r) + \beta_I P_r(x_I) = 0$ and we immediately obtain

$$\beta_r = \text{const} \prod_{i=1, i\neq r}^{N+1} (x_r - x_i)^{-1}, \tag{28}$$

which is a solution because the system (27) forms a basis of the linear space of $N-1$ degree polynomials. So for $I = N+1$ we have a unique solution (up to an arbitrary nonzero constant) given by (28).

Returning to (25) we first consider $N = 2n$, n a positive integer. Fix $I = N+1 = 2n+1$. The degree K becomes $3n+1$. It is convenient to use a new index $k = |3\kappa+1|$, where $|\kappa| \leq n$. Equation (25) can be rewritten as

$$\sum_{\kappa=-n}^{n} \beta_\kappa (3\kappa+1)^{2(j-1)} = 0, \qquad j = 1, 2, \ldots, N, \tag{29}$$

where we use new unknowns $\beta_\kappa = (-1)^\kappa c_{|3\kappa+1|}|3\kappa+1|$ instead of c_k. Using (28) we obtain

$$\beta_\kappa = \text{const} \prod_{\rho=-n, \rho\neq\kappa}^{n} ((3\kappa+1)^2 - (3\rho+1)^2)^{-1}. \tag{30}$$

We can rewrite this using binomial coefficients as

$$\beta_\kappa = \text{const}' \, (\kappa + \frac{1}{3}) \binom{2n+\frac{2}{3}}{n-\kappa} \binom{2n-\frac{2}{3}}{n+\kappa}. \tag{31}$$

The old system of unknowns is given (up to an arbitrary constant) by

$$c_{3\kappa+1} = (-1)^\kappa \, \text{sgn} \, (\kappa + \frac{1}{3}) \binom{2n+\frac{2}{3}}{n-\kappa} \binom{2n-\frac{2}{3}}{n+\kappa} \tag{32}$$

and using (22) we obtain the function $f(v)$,

$$f(v) = \sum_{\kappa=-n}^{n} (-1)^\kappa \binom{2n+\frac{2}{3}}{n-\kappa} \binom{2n-\frac{2}{3}}{n+\kappa} \sin(3\kappa+1)v. \tag{33}$$

We recall that the solution of Baxter's T-Q equation for $T(u) = T_0(u)$ is given by

$$Q_0(u) = \frac{f(2u)}{\sin 2u \cos^{2N} u} \tag{34}$$

and its zeros $\{u_k\}$ satisfy the BAE (4).

IV. Distribution of zeros of $Q_0(u)$

Function $f(v)$ (33) can be shown to satisfy a simple second order linear differential equation. The coefficient functions of this equation are closely connected with the density function of the zeros of $Q_0(u)$ in the thermodynamic limit.

Let us introduce $x = \exp(3iv)$ and rewrite $f(v)$ as

$$f(v) = F(x) = F^+(x) - F^-(x), \qquad F^+(x) = \sum_{\kappa=-n}^{n} a_\kappa x^{\kappa+\frac{1}{3}}, \qquad F^-(x) = F^+(1/x). \tag{35}$$

We have a multiplicative recursion relation for a_κ

$$\frac{a_{\kappa+1}}{a_\kappa} = -\frac{(n-\kappa)(n-\kappa-2/3)}{(n+\kappa+1)(n+\kappa+5/3)} \tag{36}$$

which gives [1]

$$\sum_{\kappa=-n}^{n} \{a_{\kappa+1}(n+\kappa+1)(n+\kappa+5/3) + a_\kappa(n-\kappa)(n-\kappa-2/3)\}x^{\kappa+\frac{1}{3}} = 0 \tag{37}$$

or

$$\sum_{\kappa=-n}^{n} a_\kappa\{(n+\kappa+1/3)^2 - 1/9\}x^{\kappa-\frac{2}{3}} + \sum_{\kappa=-n}^{n} a_\kappa\{(n-\kappa-1/3)^2 - 1/9\}x^{\kappa+\frac{1}{3}} = 0. \tag{38}$$

Using the operator $\theta = x\frac{d}{dx}$ we can rewrite this as

$$\{((\theta+n)^2 - 1/9)/x + (\theta-n)^2 - 1/9\}F^+ = 0. \tag{39}$$

This equation is invariant under the transformation $x \to 1/x$, so $F^-(x) = F^+(1/x)$ is also a solution for (39).

Alternatively, we know that in a neighbourhood around the singular point $x=-1$ there are two solutions of (39) which behave as $(x+1)^\alpha(1+O(x+1)+\ldots)$. Substituting this into (39), we obtain $\alpha_1 = 0$ and $\alpha_2 = 4n+1$. This method can be used as an another way to find $f(v)$.

Returning to variable v we have

$$\{\exp(-3iv)((d/dv+3in)^2+1) + ((d/dv-3in)^2+1)\}f(v) = 0. \tag{40}$$

Multiplication with $\exp(3iv/2)/\cos(3v/2)$ gives

$$\frac{d^2f}{dv^2} + 6n\tan(3v/2)\frac{df}{dv} + (1-9n^2)\,f = 0. \tag{41}$$

The zeros of $f(v)$, the density of which is important in the thermodynamic limit, are located on the imaginary axis in the complex v-plane. So it is convenient to make the change of variable $v = is$. It is also useful to introduce another function $g(s) = f(is)/\cosh^{2n}(3s/2)$. The differential equation for $g(s)$ is then

$$g'' + \left(\frac{9n(2n+1)}{2\cosh^2(3s/2)} - 1\right)g = 0. \tag{42}$$

Let $g(s_0) = 0$. For large n we have in a small vicinity of s_0

$$g'' + \left(\frac{3n}{\cosh(3s_0/2)}\right)^2 g = 0. \tag{43}$$

[1] $F^+(\kappa) = F(-2n, 2/3-2n, 5/3, -x)x^{1/3-n}$, where $F(a,b,c,x)$ is the usual Gauss hypergeometric function.

This equation describes a harmonic oscillator with frequency $\omega_0 = 3n/\cosh(3s_0/2)$. The distance between nearest zeros is approximately $\Delta s = \pi/\omega$ and we obtain the following density function which describes the number of zeros per unit length

$$\rho(s) = 1/\Delta s = \omega/\pi = \frac{3n}{\pi \cosh(3s/2)}. \tag{44}$$

It is possible to use the theory of Sturm-Liouville operators for a more careful consideration of (42). We note that this equation has a history as rich as the BAE. Eckart [13] used the Schrodinger equation with a bell-shaped potential $V(r) = -G/\cosh^2 r$ for phenomenological studies in atomic and molecular physics. Later it was used in chemistry, biophysics and astrophysics, just to name a few. For more recent references see for example [14].

V. η-dependence of the ground state energy of the finite XXZ spin chain

We recall Baxter's t-Q equation in terms of the second solution P,

$$t(u)P(u) = \phi(u+\eta/2)P(u-\eta) + \phi(u-\eta/2)P(u+\eta) \tag{45}$$

$P(u)$ is characterised by having degree $N+2$. As in the case of $Q(u)$ we set up a function $f(v) = \sin v\, \cos^{2N}(v/2)\, P(v/2)$, and write

$$f(v) = \sum_{k=1}^{K} c_k \sin kv, \tag{46}$$

where K is now $3n+2$. The number of equations for coefficients c_k is determined by $t(u)$ and is N as before. Thus in this case we have freedom in two of the c_k.

Solving equation (25) with this extra freedom, we obtain $f(v) = \alpha f_P(v) + \beta f_Q(v)$ where α, β are arbitrary, $f_Q(v)$ is the solution for $Q(u)$ given by (33) and

$$f_P(v) = \sum_{\kappa=-n}^{n} (-1)^\kappa \binom{2n+\frac{4}{3}}{n-\kappa} \binom{2n-\frac{4}{3}}{n+\kappa} \sin(3\kappa+2)v. \tag{47}$$

Baxter's equation can be interpreted as a discrete version of a second order differential equation, so we can express its coefficients in terms of the two independent solutions [7, 8]:

$$\phi(u) = P(u+\eta/2)Q(u-\eta/2) - P(u-\eta/2)Q(u+\eta/2) \;\equiv\; p_+q_- - p_-q_+ \tag{48}$$

and

$$t(u) = P(u+\eta)Q(u-\eta) - P(u-\eta)Q(u+\eta) \;\equiv\; P_+Q_- - P_-Q_+. \tag{49}$$

Consider now the functions t, Q and P as functions of two variables u and η. $\phi(u)$ does not depend on η, so taking the derivative w.r.t. η from (48) we obtain

$$0 = \delta p_+ q_- + p_+ \delta q_- - \delta p_- q_+ - p_- \delta q_+ + \frac{1}{2}(p'_+ q_- - p_+ q'_- + p'_- q_+ - p_- q'_+). \tag{50}$$

We use δ for the derivative w.r.t. η and $'$ for the derivative w.r.t. u. Equation (49) leads to

$$\delta t = \delta P_+ Q_- + P_+ \delta Q_- - \delta P_- Q_+ - P_- \delta Q_+ + P'_+ Q_- - P_+ Q'_- + P'_- Q_+ - P_- Q'_+ \tag{51}$$

For $\eta = \pi/3$ we have, for example, $p_+ \equiv P(u + \pi/6) = P(u + \pi/2 - \pi/3)$, so if we shift the variable u in (50) by $\pi/2$ we get

$$0 = \delta P_- Q_+ + P_- \delta Q_+ - \delta P_+ Q_- - P_+ \delta Q_- + \frac{1}{2}(P'_- Q_+ - P_+ Q'_- + P'_+ Q_- - P_- Q'_+) \tag{52}$$

Adding this equation to (51) we arrive at the formula for the derivative of the largest eigenvalue of the transfer matrix

$$\delta t \equiv \left. \frac{\partial t}{\partial \eta} \right|_{\eta = \pi/3} = \frac{3}{2}(P'_+ Q_- - P_+ Q'_- + P'_- Q_+ - P_- Q'_+) \tag{53}$$

One can use (33) and (47) to get an explicit expression.

Note that "Wronskian" (48) reduces to a complicated combinatorial identity

$$A(-K-1) - A(K-1) = (-1)^K \frac{K}{2n+1} \binom{2n - \frac{2}{3}}{2n} \binom{2n - \frac{4}{3}}{2n} \binom{4n+2}{K+2n+1}, \tag{54}$$

where K is an integer $1 - 2n \leq K \leq 2n - 1$ and

$$A(K) \equiv \sum_{\kappa=-n}^{n} \sum_{\tilde{\kappa}=-n}^{n} \binom{2n + \frac{2}{3}}{n - \kappa} \binom{2n - \frac{2}{3}}{n + \kappa} \binom{2n + \frac{4}{3}}{n - \tilde{\kappa}} \binom{2n - \frac{4}{3}}{n + \tilde{\kappa}} \delta_{\kappa + \tilde{\kappa}, K}. \tag{55}$$

VI. Finite XXZ spin chain with periodic boundary conditions and with additional magnetic field

The assymetric 6-vertex model under consideration was investigated by Baxter [22]. The associated spin chain Hamiltonian can be found in the paper of Perk and Schultz [23]. Baxter's T-Q equation with additional field can be written as follows

$$T(u)\, Q(u) = \exp(h) \sin^N(u - \eta/2)\, Q(u + \eta) + \exp(-h) \sin^N(u + \eta/2)\, Q(u - \eta). \tag{56}$$

Using a similar procedure to Section II (see also Baxter's original paper [16]) we find that for $\eta = 2\pi/3$ or for $\eta = \pi/3$ we again have a simple solution for a special choice of the field. Let us fix $\eta = 2\pi/3$ and $h = i\eta$. For this case the transfer matrix T has an eigenvalue $-\sin^n u$.

Using the method of Section III, we find that

$$\begin{aligned} f(u) \equiv \sin^N u\, Q(u) = \sum_{\kappa=0}^{n} \binom{n - \frac{1}{3}}{\kappa} \binom{n - \frac{2}{3}}{n - \kappa} \exp(iu(6\kappa - 3n)) \quad - \\ - \sum_{\kappa=0}^{n-1} \binom{n - \frac{1}{3}}{\kappa} \binom{n - \frac{2}{3}}{n - 1 - \kappa} \exp(iu(-6\kappa + 3n - 2)) \end{aligned} \tag{57}$$

where $n = N/2$.

The trigonometric polynomial $P(u) \equiv Q(-u)$ satisfies a conjugate equation which can be obtained from (56) by changing the sign of parameter h. We easily obtain (see for example [7]) the analogues of (48) and (49):

$$\phi(u) = \exp(-h)P(u+\eta/2)Q(u-\eta/2) - \exp(h)P(u-\eta/2)Q(u+\eta/2), \tag{58}$$

and

$$T(u) = \exp(-2h)P(u+\eta)Q(u-\eta) - \exp(2h)P(u-\eta)Q(u+\eta). \tag{59}$$

Using similar procedure to the previous section we find the derivative of eigenvalue $T(u)$ w.r.t. parameter h, which is the "magnetic moment" we refered to in the abstract.

$$\left.\frac{\partial T}{\partial h}\right|_{h=2i\pi/3} = -3\{\exp(h)\,Q(-u-\eta)Q(u-\eta) + \exp(-h)\,Q(-u+\eta)Q(u+\eta)\}. \tag{60}$$

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References

[1] F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter and G. R. W. Quispel, *J. Phys.* A20 (1987), 6397

[2] V. Pasqier and H. Saleur, *Nucl. Phys.* B330 (1990), 523

[3] E. K. Sklyanin, *J. Phys.* A21 (1988), 2375

[4] I. Cherednik, *Theor. Math. Phys.* 61 (1984), 977

[5] Y. K. Zhou, *Nucl. Phys.* B453 (1995), 619

[6] L. Mezincescu and R. I. Nepomichie, *Nucl. Phys.* B372 (1992), 597

[7] V. V. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov, *Commun. Math. Phys.* 177 (1996), 381; 190 (1997), 247; 200 (1999), 297

[8] G. P. Pronko and Yu. G. Stroganov, *J. Phys.* A32 (1999), 2333

[9] A. A. Belavin and Yu. G. Stroganov, *Phys. Lett.* B466 (1999), 281

[10] R. J. Baxter and P. A. Pearce, *J. Phys.* A15 (1982), 897

[11] G. E. Andrews, R. J. Baxter and P. J. Forrester, *J. of Stat. Phys.* 35 (1984), 193

[12] R. E. Behrend, P. A. Pearce and D. L. O'Brien, *J. of Stat. Phys.* 84 (1996) 1

[13] C. Eckart, *Phys. Rev.* 35 (1930), 1303

[14] M. Znojil, *New set of exactly solvable complex potentials giving the real energies, quant-ph/9912079*

[15] V. Fridkin, Yu. G. Stroganov and Don Zagier, *J. Phys.* A33 (2000), L??

[16] R. J. Baxter, *Adv. Stud. in Pure Math.* 19 (1989), 95

[17] R. J. Baxter, *Ann. Phys.* 70 (1972), 323

[18] M. N. Barber and M. T. Batchelor, *Int. J. of Mod. Phys.* B4 (1990), 953

[19] C. J. Hamer, *J. Phys.* A19 (1986) 3335

[20] Yu. G. Stroganov, to be published

[21] P. A. Pearce, *Int. J. of Mod. Phys.* A7, Suppl. 1B (1992), 791

[22] R. J. Baxter, *Stud. Appl. Math. (MIT)* 50 (1971), 51

[23] J. H. H. Perk and C. L. Schult, *Phys. Lett.* A84 (1981), 407